J. Fluid Mech. (1974), *vol.* 62, *part* 4, *pp.* 657–675
Printed in Great Britain

On the Poincaré problem for a compressible medium

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(Received 15 June 1973)

By the 'Poincaré problem' is meant the determination of the free oscillations of a contained rotating fluid, its velocity being linearized around a state of solid rotation. Compressibility requires one to introduce a basic thermodynamic profile as well as a basic velocity distribution. Here the temperature gradient has been supposed proportional to the adiabatic gradient, by introduction of a proportionality constant α ($\alpha = 0$ in the isothermal case; $\alpha = 1$ in the adiabatic case). In this formulation the system is reducible to a single second-order ordinary differential equation and its boundary condition.

It is proved that if $\alpha = 1$ the oscillation frequencies in the rotating system cannot equal plus or minus twice the rotation frequency. The negative case is pathological in the sense that there are solutions arbitrarily near the forbidden solution, and a solution curve of frequency as a function of rotation rate crosses the forbidden frequency.

The basic system is expanded in terms of a power series in $\gamma - 1$, where γ is the ratio of specific heats. The zeroth-order set of equations is solved in terms of confluent hypergeometric functions, and a solvability condition on the first-order set gives frequency shifts as functions of α. Several zeroth-order frequencies have been calculated, together with four first-order frequency shifts.

1. Introduction

The phase 'Poincaré problem' is short for 'finding the free modes of oscillation of a contained rotating fluid'. When the fluid is incompressible this search leads to the Poincaré equation, hence the name. When the fluid is compressible the problem leads to a considerably more complicated equation; approaches to solving this equation form the main subject of this paper.

One would hope that these results would have some astrophysical applications, and naturally, they are relevant to high-speed rotating fluid machines.

Of more basic interest is the question of the distinction between compressible and incompressible phenomena. In the present case one can ask whether 'acoustic' and 'inertial' waves are distinguishable when the Mach number is greater than or of order unity. This bears on the general question of how far one can push the Boussinesq equations if one is truly interested in non-acoustic phenomena at finite Mach numbers. I shall show that there is partial separation if the basic thermodynamic state is adiabatic.

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In an effort to derive general equations, the basic temperature gradient is taken to be proportional to the adiabatic gradient with the proportionality constant a parameter of the problem, allowing variation between isothermal and adiabatic states.

The plan of the paper is as follows. In § 2 the basic equations are derived in an inertial co-ordinate system, independently of the container geometry, and a simple demonstration that the eigenvalues are real is given. In § 3 the system is reduced to a single ordinary differential equation and its associated boundary condition for cylindrical geometry. The nature of this equation is explored in terms of its singular points, and it is shown to be tractable only if the ratio of the specific heats of the gas is unity. This will form the basis for an expansion explored in § 5. First, however, I show in § 4 that $|\Lambda| \neq 2\Omega$ if the basic temperature gradient is adiabatic. (Λ is the dimensional frequency in the rotating co-ordinate system and Ω the rotation rate.) This is an extension of the well-known situation in the incompressible case. The case $\Lambda = -2\Omega$ proves to be illusory. While Λ cannot equal -2Ω, it can be arbitrarily close, and a curve of $\Lambda(\Omega)$ will cross the $\Lambda = -2\Omega$ line.

In § 5 an expansion procedure in powers of $\gamma - 1$ is given, and an expression for the $O(\gamma-1)$ correction to the eigenvalue is obtained. In § 6 some actual eigenvalues are obtained for the small $\gamma - 1$ problem and a discussion is given.

2. Formulation

In deriving the perturbation equations for the usual Poincaré problem a basic velocity field $\mathbf{v} = \mathbf{\Omega} \times \mathbf{r}$ is imposed, and the equations of mass and momentum conservation are linearized about the basic state of solid rotation. In the compressible case this is inadequate; one basic thermodynamic profile must be prescribed as well. In this paper the equations will be linearized about a velocity distribution $\mathbf{v} = \mathbf{\Omega} \times \mathbf{r}$ together with a basic temperature distribution $T_0(\varpi)$. Here ϖ is the radial co-ordinate in a (ϖ, ϕ, z) co-ordinate system with $\hat{\mathbf{z}}$ parallel to $\mathbf{\Omega}$. Associated with this temperature distribution will be an entropy distribution $S_0(\varpi)$. It will be convenient to use both temperature and entropy during the analysis. The reader is reminded that these cannot be independently specified.

The basic equations in terms of P, the pressure, ρ, the density, $\mathbf{v}$, the velocity, T, the temperature, and S, the entropy, written in an inertial co-ordinate system, are

$$\rho D\mathbf{v}/Dt + \nabla P = \rho\nu\{\tfrac{4}{3}\nabla\nabla . \mathbf{v} - \nabla \times \nabla \times \mathbf{v}\}, \tag{2.1a}$$

$$D\rho/Dt + \rho\nabla . \mathbf{v} = 0, \tag{2.1b}$$

$$DS/Dt = 0, \quad P = P(\rho, S), \tag{2.1c,d}$$

where $\rho\nu$ is the coefficient of viscosity. The conditions under which viscosity may be neglected will be discussed below. D/Dt is the usual convective derivative.

If the basic quantities P_0, ρ_0, S_0 and T_0 are functions of ϖ only, the second and third equations are automatically satisfied and the basic state need only satisfy the pair

$$P_0' = \Omega^2\varpi\rho_0, \quad P_0 = P(\rho_0, S_0), \tag{2.2a,b}$$

where S_0 is specified. A prime is used to denote differentiation with respect to ϖ. By assuming a perfect gas, the pressure and temperature distributions can be related using the supplementary equation of state

$$P_0 = R\rho_0 T_0, \tag{2.2c}$$

where R is the gas constant.

The entropy and temperature distributions can be related by differentiating both equations of state with respect to ϖ and substituting from (2.2a) to eliminate the pressure. This manipulation produces the pair

$$\left.\begin{aligned} \Omega^2\varpi\rho_0 &= (\partial P_0/\partial\rho_0)_S\rho_0' + (\partial P_0/\partial S_0)_\rho S_0', \\ \Omega^2\varpi\rho_0 &= RT_0\rho_0' + R\rho_0 T_0'. \end{aligned}\right\} \tag{2.3}$$

Since $(\partial P_0/\partial\rho_0)_S$ is the adiabatic sound speed, equal to γRT_0, where γ is the ratio of specific heats, and $(\partial P_0/\partial S_0)_\rho$ can be shown to be equal to $(\gamma-1)/\gamma\rho_0 T_0$ one can rewrite this pair as

$$\left.\begin{aligned} \Omega^2\varpi\rho_0 &= \gamma RT_0\rho_0' + (\gamma-1)/\gamma\rho_0 T_0 S_0', \\ \Omega^2\varpi\rho_0 &= RT_0\rho_0' + R\rho_0 T_0', \end{aligned}\right\} \tag{2.4}$$

and subtract to obtain a relation between T_0' and S_0'. In particular, the isothermal entropy distribution is given by

$$S_0' = -\gamma R\rho_0'/\rho_0, \tag{2.5}$$

and the isentropic (adiabatic) temperature distribution by

$$T_0'/T_0 = (\gamma-1)\rho_0'/\rho_0. \tag{2.6}$$

The temperature distribution can be varied by defining α as the (constant) ratio of the actual temperature gradient to the adiabatic temperature gradient. After some manipulation it can be seen that

$$T_0(\varpi) = T_c + \frac{1}{2}\frac{\alpha(\gamma-1)}{1+\alpha(\gamma-1)}\frac{\Omega^2\varpi^2}{R}, \tag{2.7}$$

and

$$S_0'(\varpi) = \frac{(\alpha-1)\gamma}{1+\alpha(\gamma-1)}\frac{\Omega^2\varpi}{T_0}. \tag{2.8}$$

T_c is the temperature at $\varpi = 0$.

The parameter α has been introduced to establish a one-parameter family of temperature profiles including the two important special cases, the adiabatic, $\alpha = 1$, and isothermal, $\alpha = 0$, cases. It will be established below that the system is stable for $0 \leqslant \alpha \leqslant 1$.

The perturbation equations are obtained by the substitutions

$$\left.\begin{aligned} P &= P_0 + \epsilon P_1, & \rho &= \rho_0 + \epsilon\rho_1, \\ \mathbf{v} &= \mathbf{v}_0 + \epsilon\mathbf{v}_1, & S &= S_0 + \epsilon S_1, \end{aligned}\right\} \tag{2.9}$$

and retention of the linear terms. The result is

$$\rho_0\left[\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\phi}\right)\mathbf{v}_1 + 2\boldsymbol{\Omega}\times\mathbf{v}_1\right] - \rho_1\Omega^2\varpi\hat{\boldsymbol{\varpi}} + \nabla P_1 = \rho\nu[\tfrac{4}{3}\nabla\nabla\cdot\mathbf{v}_1 - \nabla\times\nabla\times\mathbf{v}_1], \tag{2.10a}$$

$$\left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\phi}\right)\rho_1+\nabla.(\rho_0\mathbf{v}_1)=0, \tag{2.10b}$$

$$\left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\phi}\right)S_1+\mathbf{v}_1.\hat{\boldsymbol{\varpi}}S_0'=0, \tag{2.10c}$$

$$\left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\phi}\right)P_1+\mathbf{v}_1.\hat{\boldsymbol{\varpi}}P_0'=\gamma RT_0\left\{\left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\phi}\right)\rho_1+\mathbf{v}_1.\hat{\boldsymbol{\varpi}}\rho_0'\right\}. \tag{2.10d}$$

The term $\Omega\partial/\partial\phi$ in the various operators arises because the equations have been written in an inertial co-ordinate system.

It is convenient to non-dimensionalize in terms of L, a length scale, M, a mass scale, and Ω and T_c according to the scheme

$$\left.\begin{array}{llll}\rho=(M/L^3)\bar{\rho}, & P=\Omega^2(M/L)\bar{P}, & \mathbf{v}=\Omega L\bar{\mathbf{u}}, \\ S=\gamma R\bar{S}, & T=T_c\bar{T}, & \mathbf{r}=L\bar{\mathbf{r}}, & t=\Omega^{-1}\bar{t}.\end{array}\right\} \tag{2.11}$$

Two dimensionless numbers are introduced by this scaling:

$$\mu^2=\Omega^2L^2/\gamma RT_c,\quad E=(\rho\nu)L/M\Omega. \tag{2.12}$$

The former is the square of a Mach number, relating the peripheral velocity to the sound speed on the axis ($\varpi=0$), and the latter an Ekman number. At this point it will be supposed that $E\ll 1$ and the viscous term discarded. This limit requires a reduction of the velocity boundary condition from $\mathbf{v}=0$ to $\bar{\mathbf{u}}.\mathbf{n}=0$. The equations obtained are

$$\bar{\rho}_0\mathscr{L}\bar{\mathbf{u}}_1-\varpi\hat{\boldsymbol{\varpi}}\bar{\rho}_1+\nabla\bar{P}_1=0, \tag{2.10a$'$}$$

$$(\partial_t+\partial_\phi)\bar{\rho}_1+\nabla.(\bar{\rho}_0\bar{\mathbf{u}}_1)=0, \tag{2.10b$'$}$$

$$(\partial_t+\partial_\phi)\bar{S}_1+\bar{u}_1\bar{S}_0'=0, \tag{2.10c$'$}$$

$$(\partial_t+\partial_\phi)\bar{P}_1+\bar{u}_1\bar{P}_0'=\mu^{-2}c^2[(\partial_t+\partial_\phi)\bar{\rho}_1+\bar{u}_1\bar{\rho}_0'], \tag{2.10d$'$}$$

where $\mathscr{L}=\partial+\partial_\phi+2\hat{\mathbf{z}}\times$, $\bar{u}_1$ is the radial component of $\bar{\mathbf{u}}_1$, a prime denotes differentiation with respect to ϖ, and

$$c^2=1+\tfrac{1}{2}\mu^2\frac{\alpha(\gamma-1)\gamma}{1+\alpha(\gamma-1)}\varpi^2 \tag{2.13}$$

is the square of a dimensionless sound speed.

Solutions proportional to $\exp[i(\sigma t+m\phi+kz)]$ will be sought, and the combination $\sigma+m$ will be called λ. (This represents the time derivative in a rotating co-ordinate system and corresponds to the usual λ in the Poincaré problem.) Equation (2.10d)$'$ is used to eliminate ρ_1 in terms of P_1. (The overbars will henceforth be dropped.) Equation (2.10c)$'$ merely defines S_1 and does not affect the dynamics. After some manipulation the pair of equations

$$i\lambda\rho_0\mathbf{u}_1+2\rho_0\hat{\mathbf{z}}\times\mathbf{u}_1-\hat{\boldsymbol{\varpi}}\varpi\frac{\mu^2}{c^2}P_1+\hat{\boldsymbol{\varpi}}\frac{K}{i\lambda}\rho_0\frac{\varpi^2}{c^2}\mu^2u_1+\nabla P_1=0, \tag{2.14a}$$

$$i\lambda\frac{\mu^2}{c^2}P_1+\rho_0\frac{\mu^2}{c^2}\varpi u_1+\rho_0\nabla.\mathbf{u}_1=0 \tag{2.14b}$$

is obtained. The constant $K=[(1-\alpha)(\gamma-1)]/[1+\alpha(\gamma-1)]$.

These are the basic equations to be considered below. They are characterized by three parameters, the dynamic Mach number, the introduced thermodynamic parameter α and $\gamma-1$.

Howard & Siegmann (1969) have established that the eigenvalues λ will be real if the temperature gradient nowhere exceeds the adiabatic one, using an energy conservation principle. The same result can be obtained directly in this case using Greenspan's (1968) technique for the incompressible problem.

Multiply (2.14*a*) by $\mathbf{u}_1^*$ and integrate over the volume using the boundary condition $\mathbf{u}_1.\mathbf{n} = 0$. This produces the equation

$$i\lambda\langle\rho_0\mathbf{u}_1^*.\mathbf{u}_1\rangle + 2\langle\rho_0\mathbf{u}_1^*.\mathbf{k}\times\mathbf{u}_1\rangle - \left\langle u_1^*\frac{\varpi\mu^2}{c^2}P_1\right\rangle + \frac{K}{i\lambda}\mu^2\left\langle\rho_0\frac{\varpi^2}{c^2}u_1^*u_1\right\rangle - \langle P_1\nabla.\mathbf{u}_1^*\rangle = 0, \quad (2.15)$$

where an asterisk denotes a complex conjugate and angular brackets denote the integral over the volume. The equation can be simplified by adding

$$\langle P_1\nabla.\mathbf{u}_1^*\rangle + \left\langle P_1\frac{\mu^2}{c^2}\varpi u_1^*\right\rangle - i\lambda^*\left\langle\frac{\mu^2}{\rho_0 c^2}P_1P_1^*\right\rangle = 0, \quad (2.16)$$

obtained by multiplying (2.14*b*)* by P_1 and integrating. After transforming the Coriolis term (cf. Greenspan 1968, p. 52) the resulting eigenvalue equation can be written as

$$A^2\lambda^2 + B\lambda - KC^2 - D^2|\lambda|^2 = 0, \quad (2.17)$$

where A, B, C and D are real.

Putting $\lambda = a+ib$ and taking the imaginary part of (2.17) gives

$$b\{2aA^2+B\} = 0, \quad (2.18)$$

so that either $b = 0$ or $a = -B/2A^2$. If the latter is true then the real part of (2.17) can be solved for b. This gives

$$(A^2+D^2)b^2 = -\frac{B^2}{4A^2}\left(1+\frac{D^2}{A^2}\right) - KC^2, \quad (2.19)$$

a contradiction unless K is sufficiently negative, but K is positive on the interval

$$-(\gamma-1)^{-1} < \alpha < 1. \quad (2.20)$$

It is clear from the definition of c^2, equation (2.13), that the lower limit cannot be reached. The minimum physically meaningful value of α in the range defined by (2.20) is that for which c^2 (1) vanishes:

$$\alpha_{\min} = -1/(\gamma-1)(1+\tfrac{1}{2}\mu^2). \quad (2.21)$$

Thus Howard & Siegmann's result is obtained, and the statement above that the problem is stable for $0 \leqslant \alpha \leqslant 1$ is confirmed.

3. Reduction to a single equation

A single equation can be obtained by solving (2.14*a*) for $\mathbf{u}_1$ in terms of P_1/ρ_0 and substituting this into (2.14*b*). The result is a second-order ordinary differential equation for $Q = P_1/\rho_0$ and a boundary condition equivalent to $\mathbf{u}_1.\mathbf{n} = 0$.

The components of $\mathbf{u}_1$ are given by

$$u_1 = \frac{-i}{4-\lambda^2}\frac{1+k_1\varpi^2}{1+k_2\varpi^2}\left\{\lambda Q' + 2\frac{m}{\varpi}Q + K\lambda\frac{\mu^2\varpi}{c^2}Q\right\}, \tag{3.1a}$$

$$v_1 = \frac{1}{4-\lambda^2}\frac{1+k_1\varpi^2}{1+k_2\varpi^2}\left\{2Q' + \frac{m\lambda}{\varpi}Q - K\left(\frac{m}{\lambda}-2\right)\mu^2\frac{\varpi}{c^2}Q\right\}, \tag{3.1b}$$

$$w_1 = -(k/\lambda)\,Q. \tag{3.1c}$$

In these equations u_1, v_1 and w_1 are the cylindrical components of $\mathbf{u}_1$ and the constants

$$k_1 = \frac{1}{2}\frac{\alpha\gamma}{1-\alpha}K\mu^2, \quad k_2 = k_1 + \frac{1}{4-\lambda^2}K\mu^2; \tag{3.2}$$

c^2 is the dimensionless sound speed, equal to $1+k_1\varpi^2$.

It seems likely that the problem is inseparable in any co-ordinates other than cylindrical, so the geometry will be so restricted and the boundary condition for (3.3) below will be taken to be $u_1(1) = 0$, where u_1 is given by (3.1*a*). Potential special cases, $\lambda = 0$, ± 2 and $1+k_1$ or $1+k_2$ vanishing, will be excluded, and discussed *a posteriori*; only the differential equation obtained by equating the curly bracket in (3.3) to zero will be considered, together with the consistent boundary condition from (3.1*a*).

After considerable additional algebra a single equation in Q results:

$$\begin{aligned} 0 = \frac{-i\lambda}{4-\lambda^2}\frac{1+k_1\varpi^2}{1+k_2\varpi^2}\Bigg\{Q'' &+ \left[\frac{1}{\varpi} + \frac{\gamma\mu^2\varpi}{1+k_1\varpi^2} - \frac{2k_2\varpi}{1+k_2\varpi^2}\right]Q' \\ &- \frac{m^2}{\varpi^2}Q + \frac{Q}{1+k_1\varpi^2}\left[\xi^2(1+k_2\varpi^2) - \left(4-\lambda^2-2\frac{m}{\lambda}\right)(2k_1+\mu^2)\right. \\ &\left. + \frac{m}{\lambda}\left(\frac{m}{\lambda}-2\right)\mu^2 K\right] + \frac{Q}{1+k_2\varpi^2}\left[\left(4-\lambda^2-2\frac{m}{\lambda}\right)2k_2\right]\Bigg\}. \end{aligned} \tag{3.3}$$

Here $\xi^2 = (4-\lambda^2)\,k^2/\lambda^2$.

By examination of (3.1) and (3.3) it can be seen that the problem is unchanged if the pair (m, λ) is replaced by $(-m, -\lambda)$. Thus m can be restricted to non-negative values without loss of generality.

It can be established that $\mu^2 \to 0$ reproduces correct limiting cases. If $\mu^2 \to 0$ and $\lambda \sim 1$, corresponding to an infinite sound speed, one can set $k_1 = k_2 = 0$ in (3.1) and (3.3) and obtain

$$Q'' + \varpi^{-1}Q' + (\xi^2 - m^2/\varpi^2)\,Q = 0,$$

$$\lambda Q' + 2m\varpi^{-1}Q = 0 \quad \text{on} \quad \varpi = 1,$$

the Poincaré eigenvalue problem in a cylinder. If $\mu^2 \to 0$ and $\lambda = \nu/\mu$, $\nu \sim 1$, corresponding to vanishing rotation, then $\xi^2 \to -k^2$ and k_1 and k_2 are asymptotically equal, so that $\lambda^2(k_1-k_2) \sim 0$ and one obtains

$$Q'' + \varpi^{-1}Q' + (\nu^2 - k^2 - m^2/\varpi^2)\,Q = 0, \quad Q' = 0,$$

the usual eigenvalue problem for acoustic modes in a non-rotating cylinder.

Restrictions	Equation type	Comment
$\alpha = 0$	$[\geqslant 7, 0, 0] \to [0, 2, 1]$	Isothermal, too complicated
$\alpha = 1$	$[\geqslant 7, 0, 0] \to [0, 2, 1]$	Adiabatic, too complicated
$\gamma = 1$	$[6, 0, 0] \to [0, 1, 1]$	Confluent hypergeometric equation
$\kappa_2 = 0$	$[6, 0, 0] \to [0, 3, 0]$	Hypergeometric equation

TABLE 1

The nature of the equation is more easily seen in terms of the new variable $x = -\frac{1}{2}\mu^2\varpi^2$. If Q is replaced by $\varpi^m V(x)$, then, in terms of x and κ_1 and κ_2, defined by

$$2k_1 = \kappa_1\mu^2, \quad 2k_2 = \kappa_2\mu^2,$$

the governing equation for V is

$$V'' + \left[\frac{m+1}{x} - \frac{\gamma}{1-\kappa_1 x} + \frac{\kappa_2}{1-\kappa_2 x}\right] V' - \frac{1}{2}\frac{V}{x(1-\kappa_1 x)}\left[\frac{\xi^2}{\mu^2}(1-\kappa_2 x) - \left(4-\lambda^2-2\frac{m}{\lambda}\right)(1+\kappa_1) + \gamma m + \frac{m}{\lambda}\left(\frac{m}{\lambda}-2\right)K\right] - \frac{1}{2}\frac{V}{x(1-\kappa_2 x)}\left(4-\lambda^2-2\frac{m}{\lambda}-m\right)\kappa_2 = 0. \quad (3.4)$$

In (3.4) it can be seen that $x = 0$, κ_1^{-1} and κ_2^{-1} are regular singular points with exponent differences m, $[\alpha(\gamma-1)]^{-1}$ and 2 respectively. The point at infinity is also singular, and it is irregular unless $\xi^2\kappa_2 = 0$.

Equation (3.4) is in general far too complex to be solved in terms of the usual tabulated functions. In the notation of Ince (1956, chap. 20) the equation is derived from a confluence of at least nine elementary singularities (two for each regular singular point and at least three for the irregular singular point). This would be symbolized by $[9,0,0]$; Ince's classification continues only as far as $[6,0,0]$, containing a generalized Lamé equation, the hypergeometric and confluent hypergeometric equations, the Mathieu and associated Mathieu equations and the Weber equation. The bracket symbolism $[p,q,r]$ represents an equation by its total number of elementary singularities p, regular non-elementary singularities q and irregular singularities r. Equation (3.4) would be written as $[0,3,1]$.

It should be clear that, in order to calculate eigenvalues or to use known asymptotic results, it will be necessary to reduce (3.4) to an equation derivable from $[p,0,0]$ where $p \leqslant 6$. Various possible parameter settings and the resulting equations are given in table 1.

A comment on table 1 is in order. The species of an irregular singular point is equal to the number of elementary singularities it contains minus two. In the third line $[0,1,1]$ could be written as $[0,1,1_2]$ to indicate an irregular singularity of species 2. It is possible to determine the species of the irregular singular point at infinity in general. I have not done so since equations containing irregular singularities are too complex even if the species is unity. Because of this, the 7 indicated for $\alpha = 0, 1$ is a minimum value; things could be worse.

It should be noted that the boundary condition under this co-ordinate transformation is converted to

$$V' = \left[\frac{m}{\lambda\mu^2}(\lambda+2)+\frac{K}{1-\kappa_1 x}\right]V \quad \text{at} \quad x = -\tfrac{1}{2}\mu^2. \tag{3.5}$$

The case $\kappa_2 = 0$ specifies a relation among μ, λ and α, so that each solution set would correspond to a different temperature distribution. Any solutions thus obtained could not be correlated unambiguously among themselves; this simplification is not useful.

The case $\gamma = 1$ is much more attractive. If ϵ is defined as $\gamma - 1$, then it is easy to see that the equation and boundary condition have coefficients which are analytic functions of ϵ, so that the solutions are analytic functions of ϵ and possess Taylor series in ϵ. The $\gamma = 1$ solution is then just the leading term in the relevant Taylor series, and for small ϵ, a physically meaningful situation, should be a useful approximation to the correct solution. This approximation also has the feature that it is independent of α; a first solution does not depend on the exact nature of the temperature profile.

To find the correction terms, it proves necessary to use a modified regular perturbation scheme (in the sense of Millman & Keller 1969) expanding λ as well as V in powers of ϵ. The formalism will be developed in §5 below. First the question of $|\lambda| = 2$ will be considered.

4. When can $|\lambda| = 2$?

The possibility of $|\lambda| = 2$ is of greatest interest. These eigenvalues are forbidden in the incompressible case, but in the general case typified by (3.1) and (3.3) there appears to be no real problem when $|\lambda| = 2$, unless $\alpha = 1$, implying adiabaticity of the basic state. (In particular (3.3) takes different forms depending on the order in which α is set equal to unity and $|\lambda|$ to 2.)

To investigate the case $\alpha = 1$ properly it is necessary to return to (2.14) and set $\lambda = \pm 2$, $\alpha = 1$ and treat μ^2 as an eigenvalue.

The components of (2.14a) are (dropping the subscript one):

$$\left.\begin{aligned} \pm 2i(\rho_0 u) - 2(\rho_0 v) - \varpi\mu^2 c^{-2} P + P' &= 0,\\ 2(\rho_0 u) \pm 2i(\rho_0 v) + im\varpi^{-1}P &= 0,\\ \pm 2i(\rho_0 w) + ikP &= 0. \end{aligned}\right\} \tag{4.1}$$

Viewed as an algebraic set for $\rho_0 \mathbf{u}$ these are singular unless

$$-\varpi\mu^2 c^{-2}P + P' = \mp m\varpi^{-1}P, \tag{4.2}$$

from which

$$P \propto \rho_0 \varpi^{\mp m}. \tag{4.3}$$

The upper sign ($\lambda = +2$) implies that the proportionality constant must equal zero, and hence that $w = 0$. The redundant pair requires that $v = iu$, and the continuity equation reduces to

$$\partial(\rho_0 u/\varpi^m)\,\partial\varpi = 0, \tag{4.4}$$

the solution to which is $$u \propto \varpi^m/\rho_0, \tag{4.5}$$
which cannot satisfy the boundary condition $u(1) = 0$; ρ_0 is inherently positive. This proves that $\lambda = +2$ is forbidden; the discussion now turns to the investigation of $\lambda = -2$.

One can again form a single equation for u:

$$\frac{\partial u}{\partial \varpi} + \left[\frac{\mu^2\varpi}{c^2} + \frac{m+1}{\varpi}\right] u = -\frac{i}{2}\left[\frac{m^2}{\varpi^2} + k^2 - 4\frac{\mu^2}{c^2}\right](P/\rho_0). \tag{4.6}$$

The homogeneous solution is $u \propto \varpi^{-(m+1)} c^{-2/(\gamma-1)}$, which is unbounded at the origin and must be discarded. From (2.4), it can be seen that $\rho_0 \propto (c^2)^{1/(\gamma-1)}$, so $P/\rho_0 \propto \varpi^m$, and the equation for u is

$$\frac{\partial(\varpi^{m+1}(c^2)^{1/(\gamma-1)} u)}{\partial \varpi} = -\frac{i}{2}\varpi^{2m+1} c^{2/(\gamma-1)}\left[\frac{m^2}{\varpi^2} + k^2 - 4\frac{\mu^2}{c^2}\right]. \tag{4.7}$$

The right-hand side can be integrated by parts. After some algebra the particular solution can be written as

$$u = U^{(m)} \sum_{n=0}^{m} (-1)^n [\tfrac{1}{2}m^2\mu^2(\gamma-1)q(q+n+1) - k^2c^2 + 4\mu^2 m(q+n+1)] \times \frac{(m-1)(m-2)\dots(m-n+1)}{q(q+1)\dots(q+n+1)}(\Pi\varpi)^{m-2n-1}, \tag{4.8}$$

where $U^{(m)}$ and Π are constants and $q = (\gamma-1)^{-1}$. This is unbounded at $\varpi = 0$. Thus it has been established that $|\lambda| = 2$ is forbidden in a cylinder if the basic temperature distribution is adiabatic.

There is one exception to the above. It can be verified that the parameters

$$m = 0, \quad \mu = \tfrac{1}{2}k, \quad \gamma = 1, \quad \lambda = -2$$

result in a family of acoustic modes characterized by

$$u = iv = 0, \quad w = ik, \quad P = \rho_0.$$

This is the simple organ-pipe mode of the original non-rotating system. Note that the phase speed $\lambda/k = -1/\mu$, which can be dimensionalized to give

$$c_p = (\gamma R T_c)^{\frac{1}{2}}.$$

The organ-pipe mode is unaffected by rotation if $\gamma = 1$.

The proof that $\lambda \neq -2$ as given fails when $\gamma = 1$, and since this case will be used below as the leading term of an expansion, it is necessary to establish the proof separately.

In this case it happens that $\rho_0 \propto \exp(\frac{1}{2}\mu^2\varpi^2)$, and the expression corresponding to (4.7) is

$$\partial[\varpi^{m+1}\exp(\tfrac{1}{2}\mu^2\varpi^2)u]/\partial\varpi = \tfrac{1}{2}i\varpi^{2m+1}\exp(\tfrac{1}{2}\mu^2\varpi^2)[4\mu^2 - k^2 - m^2/\varpi^2]. \tag{4.9}$$

Integrating this expression and simplifying one finds

$$u \propto (4\mu^2 - k^2)\frac{\varpi^{m-1}}{\mu^2} - \frac{\varpi^{m-3}}{\mu^2}[\mu^2 m^2 + 2(4\mu^2 - k^2)] \times \left\{1 - \frac{2(m-1)}{\mu^2}\frac{1}{\varpi^2} + \frac{4(m-1)(m-2)}{\mu^2\varpi^2}\dots\right\}. \tag{4.10}$$

If $\mu^2 \neq 2k^2/(8+m^2)$ the series term is unbounded.

If $\mu^2 = 2k^2/(8+m^2)$, u has a single term, viz.

$$u \propto -\tfrac{1}{2}m^2\varpi^{m-1}, \tag{4.11}$$

which does not vanish at $\varpi = 1$. Thus, the statement $\lambda \neq -2$ holds for $\gamma = 1$ as well.

A mathematical proof, which says that λ cannot equal ± 2, is not the same as a physical proof. Physics demands that one establish that λ cannot be near ± 2, and it will become clear from the numerical results in §5 that λ can be arbitrarily near -2 for finite μ. The mathematics behind this is easily established from (3.3).

If $\lambda = \pm 2$ and $\alpha = 1$, equation (3.3) reduces to

$$Q'' + \left[\frac{1}{\varpi} + \frac{\mu^2\varpi}{1+k_1\varpi^2}\right]Q' + \left[\frac{\pm m\mu^2}{1+k_1\varpi^2} - \frac{m^2}{\varpi^2}\right]Q = 0, \tag{4.12}$$

omitting the (infinite) factor outside the curly brackets. If the lower sign is chosen, (4.12) admits the solution $Q = \varpi^m$, for which, referring to (3.1a), $u \equiv 0$ (again omitting the factor $(4-\lambda^2)^{-1}$).

If $u \equiv 0$, it follows from (2.14) that in fact Q must be identically zero, meaning that the mode has no pressure or velocity associated with it. However, there will be modes arbitrarily close to this which do have pressure and velocity fields associated with them; an eigenvalue curve in λ, μ space *can* cross $\lambda = -2$.

5. A perturbation scheme for $\gamma - 1 \ll 1$

Let

$$\left.\begin{aligned} V &= V_0 + \epsilon V_1 + \dots, \\ \lambda &= \lambda_0 + \epsilon\lambda_1 + \dots. \end{aligned}\right\} \tag{5.1}$$

The pair (V_0, λ_0) is obtained by solving the $\gamma = 1$ equations, which are, from (3.4) and (3.6),

$$xV_0'' + (m+1-x)\,V_0' - a_0 V_0 = 0, \tag{5.2a}$$

$$V_0' = \frac{m}{\mu^2}\frac{\lambda_0+2}{\lambda_0}V_0 \quad \text{at} \quad x = -\tfrac{1}{2}\mu^2. \tag{5.2b}$$

The solution to (5.2a) which is bounded at the origin is

$$V_0 = \Phi(a_0, m+1; x),$$

where
$$a_0 = \frac{1}{2}\left\{\frac{\xi^2}{\mu^2} + \frac{\lambda_0+2}{\lambda_0}[m + \lambda_0^2 - 2\lambda_0]\right\}$$

and the notation is that of Erdélyi *et al.* (1953). The eigenvalue relation becomes

$$\frac{m}{\lambda}(\lambda+2)\,\Phi(a, m+1; -\tfrac{1}{2}\mu^2) - \frac{\mu^2 a}{m+1}\Phi(a+1, m+2; -\tfrac{1}{2}\mu^2) = 0. \tag{5.3}$$

(In (5.3) the zero subscripts have been suppressed in the interest of clean equations. This practice will be continued below unless there is distinct danger of ambiguity.)

It should be remarked in passing that the limiting forms of (5.3) as $\mu^2 \to 0$ reproduce the appropriate incompressible formulae. If $\mu^2 \to 0$ and $\lambda \sim 1$, then

$$\Phi(a, m+1, -\tfrac{1}{2}\mu^2) \sim m!\,(\tfrac{1}{4}\mu^2/\xi^2)^{-\frac{1}{2}m} J_m(\xi) \tag{5.4}$$

and (5.3) reduces to

$$(\lambda+2)\, mJ_m(\xi) - \lambda\xi J_{m+1}(\xi) = 0. \tag{5.5}$$

If $\mu^2 \to 0$ and $\lambda = \nu/\mu$, where $\nu \sim 1$, the same limiting process yields the asymptotic eigenvalue equation

$$J'_m((\nu^2-k^2)^{\frac{1}{2}}) = 0, \tag{5.6}$$

which is the correct non-rotating acoustic limit.

Large μ^2 asymptotics are somewhat more complicated. If $\mu^2 \to \infty$ while λ behaves in such a way that a remains bounded, the asymptotic representation

$$\Phi(a, m+1; -\tfrac{1}{2}\mu^2) \sim \frac{\Gamma(m+1)}{\Gamma(m+1-a)}\left(\frac{2}{\mu^2}\right)^a$$

can be used to reduce the eigenvalue equation (5.3) to $4-\lambda^2 = 0$. Thus, $\lambda = \pm 2$ provide limiting values of λ as $\mu^2 \to \infty$. (The value -2 is suspect, and in fact is probably not a limiting value.) *It should be noted that λ must be bounded away from zero in order to keep a finite; the question of $\lambda = 0$ being an asymptote is not answered.*

The $O(\gamma-1)$ differential equation obtained by differentiating (3.4) with respect to ϵ and setting $\epsilon = 0$ can be written as

$$\begin{aligned} &xV_1'' + [(m+1)-x]\, V_1' - aV_1 \\ &\quad = \lambda_1 \frac{\mu^2\lambda^4 + \mu^2 m\lambda^2 - 4}{\mu^2\lambda^3} V_0 - \alpha\left\{xV_0' + \frac{1}{2}\left(4-\lambda^2 - 2\frac{m}{\lambda} - m\right)xV_0\right\} \\ &\quad\quad - (1-\alpha)\left\{\frac{2-\lambda^2}{4-\lambda^2}V_0' + \frac{k^2}{\lambda^2\mu^2}xV_0 - \frac{1}{2}\left[m + \frac{m}{\lambda}\left(\frac{m}{\lambda}-2\right) + 2 - \frac{m}{\lambda(2-\lambda)}\right]V_0\right\}, \end{aligned} \tag{5.7}$$

where I have suppressed the zero subscript on λ. The boundary condition, from (3.6), is

$$\begin{aligned} (\lambda+2)\, mV_1 - \lambda\mu^2 V_1' &= (\mu^2 V_0' - mV_0)\,\lambda_1 - (1-\alpha)\,\lambda\mu^2 V_0 \\ &= 2m\lambda^{-1}V_0\lambda_1 - (1-\alpha)\,\lambda\mu^2 V_0. \end{aligned} \tag{5.8}$$

The condition that the pair (5.7) and (5.8) can be solved determines λ_1. Unfortunately the standard theorems I have been able to find on the subject do not apply directly, because the operator in (5.7) is singular. The usual technique of multiplying by the solution to the adjoint problem and integrating over the interval does work but it is necessary to establish this.

Consider the model problem

$$xu'' + (\mu - x)\, u' - au = g_0(x) + \lambda_1 g_1(x), \tag{5.9}$$

subject to the boundary condition

$$u(x_0) + fu'(x_0) = \gamma_0 + \lambda_1\gamma_1, \tag{5.10}$$

and seek a solution of the form

$$u(x) = \sum_{n=0}^{\infty} A_n x^n. \tag{5.11}$$

If the functions $g_0(x)$ and $g_1(x)$ possess power series expansions

$$g_0(x) = \sum_{n=0}^{\infty} B_n x^n, \quad g_1(x) = \sum_{n=0}^{\infty} C_n x^n, \tag{5.12}$$

it can be established that

$$u = A_0 \Phi(a, \mu; x) + u_0 + \lambda_1 u_1, \tag{5.13}$$

where u_0 and u_1 depend on g_0 and g_1, respectively. Since $\Phi(a, \mu; x)$ is proportional to the zeroth-order solution, it does not contribute to the boundary condition and A_0 may be set equal to zero. Since $u_0(0) = 0 = u_1(0)$, the particular solution may be chosen to be zero at $x = 0$; this is the crucial point for which the above discussion was conducted. The proper choice of λ_1 now allows one to satisfy the boundary condition (5.10).

The following procedure produces a formula for λ_1. Multiply (5.9) by a function v and integrate between zero and x_0 to produce the equation

$$\int_0^{x_0} \bar{L}(v)\, dx = x_0 v(x_0)\, u'(x_0) - u(x_0)\,[x_0 v'(x_0) + v(x_0)] + (\mu - x_0)\, u(x_0)\, v(x_0) - \int_0^{x_0} v g(x)\, dx. \tag{5.14}$$

The right-hand side may be rearranged to give

$$\int_0^{x_0} \bar{L}(v)\, dx = \frac{x_0}{f} v(x_0)\,(\gamma_0 + \lambda_1 \gamma_1) + u(x_0)\left[\left(\mu - x_0 - 1 - \frac{x_0}{f}\right) v(x_0) - x_0 v'(x_0)\right] - \int_0^{x_0} v g(x)\, dx, \tag{5.15}$$

so that if v is a solution of the adjoint problem

$$\bar{L}(v) = (xv)'' - [(m+1-x)\, v]' - av = 0, \tag{5.16a}$$

$$[f(\mu - x_0 - 1) - x_0]\, v(x_0) - f_0 x_0 v'(x_0) = 0, \tag{5.16b}$$

then (5.15) can be used to determine λ_1. After some manipulation

$$\left\{\frac{\gamma_1 x_0}{f} v(x_0) - \int_0^{x_0} g_1(x)\, v(x)\, dx\right\} \lambda_1 = -\left\{\frac{\gamma_0 x_0}{f} v(x_0) - \int_0^{x_0} g_0(x)\, v(x)\, dx\right\}. \tag{5.17}$$

To find v, expand (5.16a):

$$xv'' - (m - 1 - x)\, v' - (a-1)\, v = 0,$$

which has the solution (Erdélyi *et al.* 1953)

$$(-x)^m e^{-x} \Phi(a, m+1; x),$$

hich satisfies (5.16b). This allows a formal solution for λ_1.

6. Discussion and some eigenvalues

The expansion procedure in §5 is a step leading to approximate curves of λ *vs.* μ. In particular the behaviour of the zeroth-order ($\gamma = 1$) eigenvalues is likely to be typical of that of the eigenvalues for small $\gamma - 1$, and the distinction between the adiabatic and isothermal cases is not likely to be crucial in understanding the general behaviour of the various modes with changing Mach number. This is based on the analyticity of the coefficients of (3.3) as a function of $\gamma - 1$.

Even the simple relation (5.2) in general defeats exact analysis. However, for special values of a_0, the confluent hypergeometric series can be written in terms of elementary functions. In particular, if $a_0 = -n$, a negative integer, the series terminates. If Kummer's transformation (Erdélyi *et al.* 1953)

$$\Phi(a_0, c; x) = e^x \Phi(c - a_0, c; -x) \tag{6.1}$$

is applied these functions will reduce to exponential functions times polynomials if $a_0 = c + n$, with n a positive integer. For integer values of a_0 not covered by these cases the solution can sometimes be simplified by inspection of the series

$$\Phi(a_0, c; x) = 1 + \frac{a_0}{c} x + \frac{a_0(a_0+1)}{c(c+1)} \frac{1}{2!} x^2 + \dots. \tag{6.2}$$

(The subscript on a_0 serves no further useful purpose and will be dropped.)

I have obtained a number of eigenvalues, which are listed in the appendix. Some have been obtained algebraically and some by iteration. There is a discussion of this in the appendix. It is only necessary here to note that the result of solving the eigenvalue relation (5.2) under the requirement that a have a fixed value produces an eigenvalue pair, a value of λ and of μ satisfying both conditions. Thus, a set of isolated points in a λ, μ space is created, and the question arises of how to draw curves through these points and how to connect them to the known asymptotic values.

Every eigenfunction is characterized by three wavenumbers. Two are the parameters m and k in the ϕ and z directions, and the third will be denoted by s, defined as one more than the number of nodes in the radial velocity *vs.* radius profile. Thus, '$s = 1$' is the 'gravest' mode in the radial direction. Alternative definitions of the radial wavenumber are possible, but this seems quite simple and natural.

At zero Mach number there are four eigenmodes corresponding to each set (s, m, k): positive and negative (λ) acoustic and inertial. These are divided from each other by the theoretical barriers at $\lambda = \pm 2$ and $\lambda = 0$. The barrier at -2, however, proves to be illusory, as indicated by the discussion at the end of §4.

For positive λ the points can be connected quite simply. All those with $\lambda > 2$ belong to the acoustic track, and those with $\lambda < 2$ belong to the inertial track. This arrangement of eigenvalue pairs reveals an additional feature that can be used to sort out the negative eigenvalues: the parameter a decreases monotonically away from $+\infty$ as the Mach number increases away from zero.

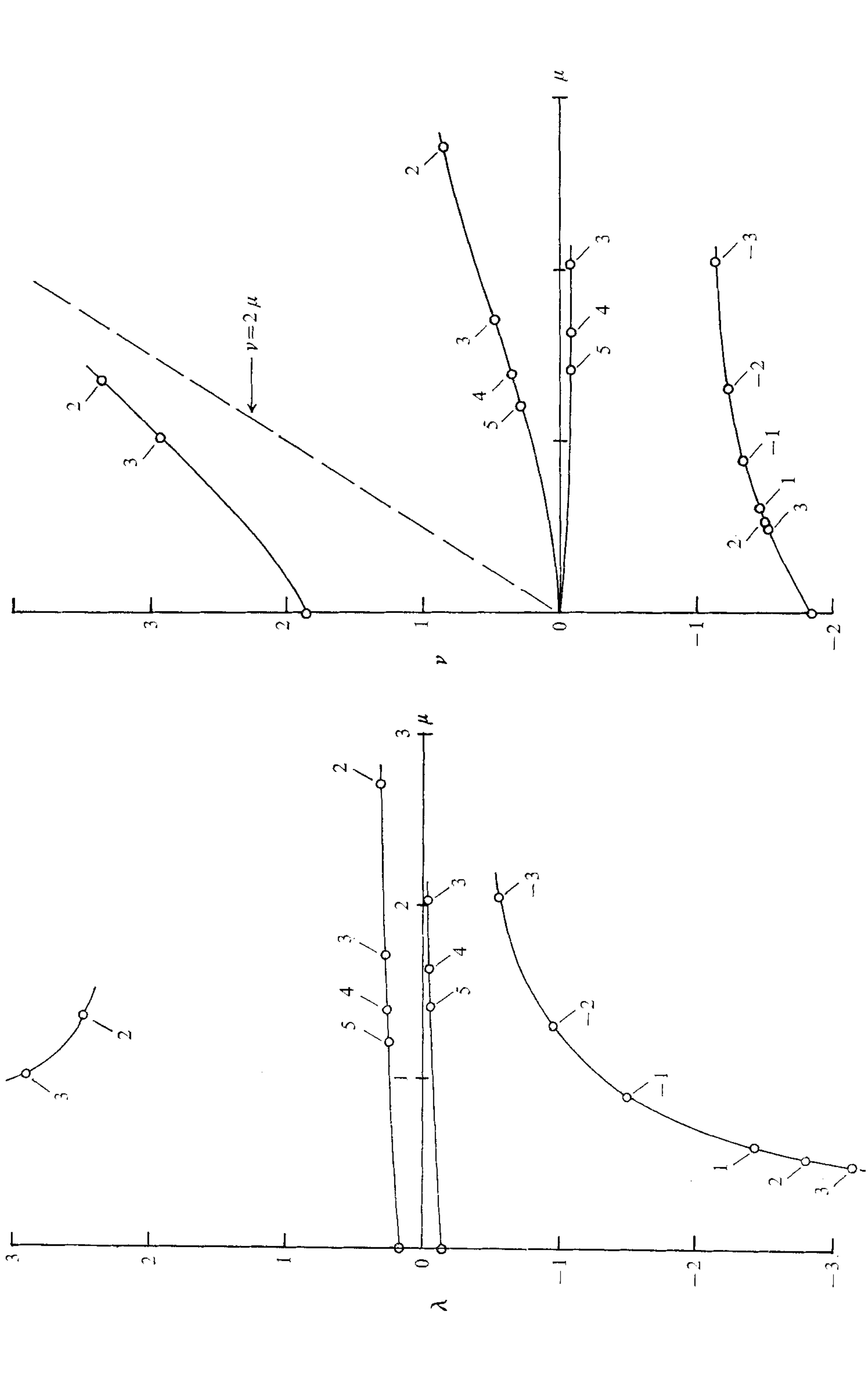

FIGURE 1. Eigenvalues in (a) the inertial representation (λ) and (b) the acoustic representation ($\nu = \mu\lambda$) as a function of μ: $(s, m, k) = (1, 1, \frac{1}{10}\pi)$.

The largest number of eigenvalues has been calculated for $(s, m, k) = (1, 1, \frac{1}{10}\pi)$, and these are plotted in figures 1 (*a*) and (*b*). The first figure gives $\lambda = \lambda(\mu)$ and the second $\nu = \nu(\mu)$. The frequency $\nu = \mu\lambda$ is that which produces finite frequencies in the limit $\mu \to 0$; it is the frequency that would have appeared had the sound speed been used as a dimensional base.

Both figures give the same general impression. There is a clear separation between inertial and acoustic modes in the positive-λ half of the figure and an apparent convergence in the negative half of the figure. The numbers on each curve represent the value of a at the point shown. Except for the curve crossing $\lambda = -2$, there is a minimum permissible value of a. This can be shown by demonstrating impossible values of a on the track in question.

If $a = 1$ the eigenvalue relation reduces to

$$\frac{2}{\lambda} = \tfrac{1}{4}\mu^4 \frac{\exp(-\frac{1}{2}\mu^2)}{1-\exp(-\frac{1}{2}\mu^2)} - 1. \tag{6.3}$$

The right-hand side has its maximum value at $\mu = 1{\cdot}785$, and the value is $-0{\cdot}3524$, requiring λ to be negative. Thus, the minimum value for a on the two positive tracks is greater than unity, if one accepts the indicated monotonic behaviour of a as μ increases.

I offer the conjecture that $a \to 1$ as $\mu \to \infty$ in both cases, with $\lambda \to +2$ in the acoustic and $\lambda \to 0$ in the inertial case.

If $a = 2$ the eigenvalue relation reduces to

$$\mu^2 = (2+\lambda)/\lambda, \tag{6.4}$$

requiring that λ be positive, or less than -2. Thus, a cannot equal 2 on the negative inertial track. It can be conjectured that $\lambda \to 0$ as $\mu \to \infty$ in such a way that $a \to 2$ on this track.

There appears to be no limit on a on the negative acoustic track. The value $a = 0$ at $\lambda = -2$ is special; there the mode must have zero amplitude. On this track, I suspect that $a \to -\infty$ as $\lambda \to 0$ and $\mu \to \infty$.

In view of the clear separation between inertial and acoustic modes, W. V. R. Malkus (personal communication) has brought up the question of internal waves, corresponding to an incompressible, stratified and 'non-rotating' basic state, where rotation would enter only to provide a 'gravitation-like' restoring force, without any Coriolis effect. A full discussion of this point is beyond the scope of the paper; I offer some thoughts on the problem in the interest of stimulation.

First it is clear that such waves cannot be discovered using the expansion scheme in § 5. Since the Brunt frequency is proportional to $(\gamma - 1)^{\frac{1}{2}}$, all the zeroth-order modes would have zero frequency. However, $\lambda = 0$ is not a solution to the zeroth-order problem, so that the modified perturbation scheme outlined cannot ever uncover these modes.

A more serious question is whether such modes can be distinguished from the inertial modes: whether they have an independent existence. One could answer yes to this question if it were possible to construct a limiting scheme which exposed this separate limit. One is in doubt because everything in the problem appears to depend on either rotation or compressibility. There is neither an independent 'gravity' nor an independent stratification.

Since the Brunt frequency is proportional to $(\gamma-1)^{\frac{1}{2}}$, one possible limit is $\gamma \to \infty$. If $\gamma \to \infty$ with $\lambda \sim 1$, equation (3.3) is transformed into a Bessel equation of order m in $\xi\varpi$; the Poincaré problem has reappeared. If $\gamma \to \infty$ with $\lambda = \nu\gamma^{\frac{1}{2}}$, things are more delicate. After some manipulation, it can be seen that the result reproduces the acoustic problem. These arguments fail if $\alpha \equiv 0$. In this case, the mathematical expression is singular. The difficulty arises from the facts that $k_1 \equiv 0$ and everything else depends on $\gamma\mu^2 = M^2$, which is independent of γ. The meaning of all this is unclear and will not be discussed further.

From this brief discussion, I offer the conjecture that there are no distinct internal waves in the sense of this discussion.

So that this paper does not dwindle off in a welter of conjecture, I shall summarize briefly the main results.

The problem of the free oscillations of a compressible fluid in uniform rotation with a quadratic radial temperature profile has been reduced to a single second-order ordinary differential equation and its associated boundary condition: equations (3.4) and (3.6). The problem has been solved in terms of Taylor series expansions of eigenfunctions and eigenvalues in powers of $\gamma-1$. The legitimacy of this process is assured by the analyticity of (3.4) and (3.6) as functions of $\gamma-1$. The lowest order eigenfunctions and eigenvalues are independent of the temperature profile, and for small $\gamma-1$, should be typical of the actual eigenvalues.

Several lowest order eigenvalue pairs have been found and are listed in the appendix. One set has been plotted graphically over Mach numbers between zero and about three, which spans most of the practical range now available. (The $\mu \to \infty$ limits have of necessity been left to conjecture.) The plot can be resolved into four separate tracks which can be unambiguously identified with known zero-Mach-number (Boussinesq) solutions. Limitations of space alone prevent the presentation of several other such curves.

The computations were supported in part by Project MAC, an M.I.T. interdepartmental laboratory sponsored by the Advanced Research Projects Agency, Department of Defense, under Office of Naval Research Contract N00014-70-A-0362-0001, and in part by the Department of Mathematics, M.I.T., under National Science Foundation Grant GP 22796. I am grateful to R. Fateman for showing me how to use his computer, and to J. Klincewicz for calculating some of the $\mu = 0$ inertial eigenvalues.

Appendix. Tabulated eigenvalues

The eigenvalue pairs given below have been calculated in a number of ways. The $\mu = 0$ acoustic modes were calculated directly using the relevant Bessel-function zeros. Some $\mu = 0$ inertial modes were calculated numerically by J. Klincewicz; others were done graphically by me. The number of significant figures indicates the difference.

Most of the eigenvalue pairs were calculated using the MACSYMA system of symbolic manipulation (Bogen 1973), which actually solved the equations and found roots using rational arithmetic accurate to a tolerance of 10^{-8}. I found a

s	m	k	a	μ	λ	ν
1	1	$\frac{1}{10}\pi$	∞	0	−0·1510	—
			5	1·414	−0·0683	−0·09658
			4	1·636	−0·0563	−0·09211
			3	2·0389029	−0·0401947	−0·08195309
			∞	0	0·17806	—
			5	1·208	0·2402	0·2902
			4	1·390	0·2549	0·3543
			3	1·7038415	0·2779246	0·4735395
			2	2·7150200	0·3139060	0·8522611
			∞	0	—	1·86779
			3	0·4874226	−3·1437981	−1·5323583
			2	0·5362710	−2·8073587	−1·5055051
			1	0·605	−2·43	−1·47
			−1	0·899062	−1·4967657	−1·3456852
			−2	1·3007228	−0·9507138	−1·2366151
			−3	2·0515594	−0·5572371	−1·1432050
			∞	0	—	1·86779
			3	1·0068501	2·891196	2·9110009
			2	1·3424202	2·4934796	3·3472974
1	1	$\frac{1}{2}\pi$	∞	0	−0·6276	—
			3	2·2485099	−0·3622256	—
			∞	0	0·99552	—
			3	1·31904205	0·8773875	—
			2	1·8850667	0·7832459	—
			∞	0	—	−2·42020
			3	0·5624680	−3·9073834	−2·1977789
			2	0·6407208	−3·3928387	−2·1738623
			1	0·76869	−2·7813	−2·1379
1	1	$\frac{1}{2}\pi$	−1	1·909805	−1·023601	−1·954878
			−2	3·6342636	−0·5204293	−1·8913772
			−3	5·47737	−0·342487	−1·87592902
			∞	0	—	2·42020
			3	0·9921496	3·1387363	3·1140961
			2	1·3290610	2·6095927	3·4683079
1	1	$\frac{5}{2}\pi$	∞	0	−1·675	—
			3	2·4276901	−1·5027396	—
			∞	0	1·99	—
			3	1·09164452	1·9223866	—
			2	1·4269889	1·929948	—
			∞	0	—	−8·06691
			3	0·7216902	−11·125969	−8·0295033
			2	0·8830585	−9·0823324	−8·0202308
			1	1·25585	−6·36933	−7·9990
			−1	11·650801	−0·679636	−7·918304
			−2	19·550793	−0·4049817	−7·9177131
			−3	27·463433	−0·28829436	−7·91755295
			∞	0	—	8·06691
			3	0·8669358	9·3322093	8·0904262
			2	1·1312060	7·1523843	8·0908200

TABLE 2 *(continued overleaf)*.

s	m	k	a	μ	λ	ν
1	2	$\frac{1}{2}\pi$	∞	0	−0·52569	—
			4	2·5704542	−0·2331754	—
			∞	0	0·66724	—
			4	1·9602584	0·7400696	—
			3	2·7143002	0·7452362	—
			∞	0	—	−3·43449
			4	0·8175386	−3·700578	−3·0253650
			3	0·8961631	−3·3419898	−2·9949679
			−1	1·880769	−1·458906	−2·743865
			−2	2·6877844	−0·9912350	−2·6592697
			∞	0	—	3·43449
			4	1·4741483	3·056201	4·5052936
			3	1·8815911	2·5967535	4·8860254
1	8	$\frac{1}{2}\pi$	∞	0	−0·251	—
			10	4·2512469	−0·0330306	—
			∞	0	0·19732	—
			10	3·7939218	1·1126944	—
			9	4.4823225	1·3232748	—
			∞	0	—	−9·77446
			10	1·9696994	−4·481057	−8·8263354
			9	2·0615202	−4·2665137	−8·7955042
			−1	4·7584	−1·7555	−8·3510
			−2	5·4165145	−1·5358847	−8·3191417
			∞	0	—	9·77446
			10	3·1851802	3·3799857	10·7658637
			9	3·6655406	2·9432388	10·788561

TABLE 2. Some eigenvalues

few additional roots by hand calculation, accurate to three or four significant figures.

First-order corrections were calculated for four eigenvalues:

$$\lambda = -3{\cdot}3928387 + [9{\cdot}945841\times 10^{-3}\alpha - 0{\cdot}84322789\,(1-\alpha)]\,(\gamma-1)+\ldots,\qquad \mu = 0{\cdot}6407208;$$

$$\lambda = -1{\cdot}07236011 + [0{\cdot}37499931\alpha + 0{\cdot}27156155(1-\alpha)]\,(\gamma-1)+\ldots,\qquad \mu = 1{\cdot}9098039;$$

$$\lambda = 0{\cdot}78324594 - [0{\cdot}3907219\alpha + 0{\cdot}06741905(1-\alpha)]\,(\gamma-1)+\ldots,\qquad \mu = 1{\cdot}8850667;$$

$$\lambda = 2{\cdot}6095927 - [0{\cdot}5506557\alpha - 1{\cdot}49747536(1-\alpha)]\,(\gamma-1)+\ldots,\qquad \mu = 1{\cdot}3290610.$$

The eigenvalue pairs of each set (s,m,k) in table 2 are arranged in groups: negative inertial, positive inertial, negative acoustic, positive acoustic. Within each group the pairs are arranged in order of decreasing a. For $(s,m,k) = (1,1,\frac{1}{10}\pi)$ both λ and $\nu = \mu\lambda$ are given for all the eigenvalues, because they have been used to draw the figures. For other (s,m,k) the ν values of inertial modes have been omitted as not meaningful.

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